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ABSTRACT

The purposes of this study were: (1) to develop a skills hierarchy for rational number multiplication using modified intraconcept analysis procedures developed by Uprichard and Phillips in two preview studies, and (2) to evaluate and modify the hierarchy using the Walbesser technique (AAAS) and pattern analysis. Subskills were delineated from two points of view: mathematical and psychological. Alternative forms of an instrument evaluating the performances on 34 identified subtasks were administered to 225 subjects in grades 5, 6, and 7. The final hierarchy, implications, and hierarchy development strategies are discussed. (Author/MP)

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An Empirically Derived Hierarchy of Intraconcept Relationships: Rational Number Multiplication

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Generation and validation of learning hierarchies has been a popular issue among many researchers since the 1960's. Uprichard and Phyllips (1976, 1977) reviewed this literature in their two intraconcept analysis studies. They reported that the studies reviewed collectively suggest that hierarchical knowledge structures exist and that optimal learning sequences can be developed. However, researchers in this area agree that optional learning sequences are difficult to validate (Gagne, 1968; Pyatte, 1969). Several validation techniques exist, but the ones which are most definitive require that enormous numbers of task permutations be taught. This results in high levels of resource expenditure (Resnick and Wang, 1969; Eisenberg and Walbesser, 1971; White, 1974). While previous studies have focused on linear sequences, the present study sought to extend the use of the Uprichard and Phillips intraconcept analysis procedure to non-linear sequences of subtasks.

Background

The usual method employed for hypothesizing learning hierarchies has been to utilize Gagne's concept of task analysis. Uprichard and Phillips (1976, 1977) devised the intraconcept analysis procedure to extend the benefits of task analysis. Intraconcept analysis differs from task analysis in that task analysis employs only logical criteria in the analysis of prerequisites whereas intraconcept analysis employed both logical and

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psychological criteria. In the two studies by Uprichard and Phillips and in the one completed by Hill (1976), the intraconcept analysis procedure yielded more variables for investigation which in turn resulted in more complete hierarchies from an instructional perspective.

Examples of intraconcept analysis variables included in the past have been prime and composite numbers in addition and subtraction with fractions (Uprichard and Phillips, 1976, 1977) and zeros in division (Hill, 1976). In the present study, an example is numerators of one and not one. Logical task analysis procedures would not yield these variables for study, but knowledge of learner performance yields these additional variables.

In the present study, the investigators used the intraconcept analysis procedure to develop a hierarchy for rational number multiplication. The investigators believe that neither this study nor any of the three studies previously completed have generated learning hierarchies; this point of view is elaborated later in the discussion section. Also, since changes were made in the originally hypothesized hierarchy as a result of preliminary analyses, the researchers also assert that neither the present hierarchy nor any of those previously developed are validated. The outcome of all the studies to date can probably best be described as generated retention hierarchies; this point of view will also be discussed later.

Procedures

Five variables were identified in the intraconcept analysis: integers, mixed numerals, improper fractions, unit fractions, and non-unit fractions. Thirty-four tasks were identified as ordered triplets for positions of these five variables in the equation factor x factor = product. The case integer x integer was not included.

Previous studies in arithmetic (Brueckner, 1929) indicate that poor performance on basic facts is a major source of errors in multiplication. To control for this, basic facts multiplication and division rated "difficult" by Murray (1939) were excluded from this study since the researchers sought to discern hierarchical relationships among subtasks independent of this variable.

Instrumentation

Sample items were constructed from the thirty-four ordered triplets; they are presented in Table 1.

Insert Table 1 About Here

Five forms were constructed to ensure that no student would readily see another's work, to control for the possibility of influencing performances based on a particular permutation of items, and to allow for a variety of multiplication and division combinations. A table of random numbers was used to determine the order of the thirty-four items on each of the five forms. These forms were piloted on 254 subjects in grades 5 and 6. The KR-20 was .93.

A second instrument, the Test of Prerequisites, was also developed. The purpose of this test was to measure students' competence in three selected prerequisites. The test contained eight multiplication combinations of moderate difficulty, eight improper fractions to be changed to mixed numerals, and four mixed numerals to be changed to improper fractions. The criteria used on this test were: 75% total score and at least 50% on each of the three subtests.

Sample

Eight classes (three fifth grade classes, three sixth and two seventh)

Table 1

RATIONAL NUMBER MULTIPLICATION PROBLEM TYPES		
Item Number	Type*	Example
1	ABC	$3 \times \frac{1}{4} = \frac{3}{4}$
2	ABE	$4 \times \frac{1}{3} = \frac{4}{3} = 1 \frac{1}{3}$
3	ACC	$2 \times \frac{3}{7} = \frac{6}{7}$
4	ACE	$5 \times \frac{2}{3} = \frac{10}{3} = 3 \frac{1}{3}$
5	ADE	$3 \times \frac{3}{2} = \frac{9}{2} = 4 \frac{1}{2}$
6	AEE	$4 \times 1\frac{1}{3} = 4 \times \frac{4}{3} = \frac{16}{3} = 5 \frac{1}{3}$
7	BAC	$\frac{1}{5} \times 2 = \frac{2}{5}$
8	BAE	$\frac{1}{3} \times 3 = \frac{3}{2} = 1 \frac{1}{2}$
9	BBB	$\frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$
10	BCC	$\frac{1}{5} \times \frac{3}{4} = \frac{3}{20}$
11	BDC	$\frac{1}{4} \times \frac{5}{3} = \frac{5}{12}$
12	BDE	$\frac{1}{3} \times \frac{7}{2} = \frac{7}{6} = 1 \frac{1}{6}$
13	BEC	$\frac{1}{3} \times 2\frac{1}{2} = \frac{1}{3} \times \frac{5}{2} = \frac{5}{6}$
14	BEE	$\frac{1}{2} \times 2\frac{1}{3} = \frac{1}{2} \times \frac{7}{3} = \frac{7}{6} = 1 \frac{1}{6}$
15	CAC	$\frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$
16	CBC	$\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$
17	CCC	$\frac{2}{5} \times \frac{2}{3} = \frac{4}{15}$

Table 1 (con't)

TEST FOR RATIONAL NUMBER MULTIPLICATION ITEM ANALYSIS STATISTICS

Item Number	Type*	Example
18	CDC	$\frac{3}{5} \times \frac{3}{2} = \frac{9}{10}$
19	CDE	$\frac{2}{3} \times \frac{5}{3} = \frac{10}{9} = 1 \frac{1}{9}$
20	CEC	$\frac{2}{3} \times 1 \frac{1}{3} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$
21	CEE	$\frac{7}{2} \times \frac{1}{3} = \frac{7}{6} = 1 \frac{1}{6}$
22	DAE	$\frac{7}{4} \times \frac{1}{2} = \frac{7}{8}$
23	DBC	$\frac{7}{3} \times \frac{1}{2} = \frac{7}{6} = 1 \frac{1}{6}$
24	DCC	$\frac{3}{2} \times \frac{3}{7} = \frac{9}{14}$
25	DCE	$\frac{5}{3} \times \frac{2}{3} = \frac{10}{9} = 1 \frac{1}{9}$
26	DDE	$\frac{3}{2} \times \frac{7}{5} = \frac{21}{10} = 2 \frac{1}{10}$
27	DEE	$\frac{3}{2} \times 1 \frac{3}{4} = \frac{3}{2} \times \frac{7}{4} = \frac{21}{8} = 2 \frac{5}{8}$
28	EAE	$2\frac{2}{3} \times 2 = \frac{8}{3} \times 2 = \frac{16}{3} = 5 \frac{1}{3}$
29	EBC	$1 \frac{1}{6} \times \frac{1}{3} = \frac{7}{6} \times \frac{1}{3} = \frac{7}{18}$
30	EBE	$2\frac{1}{3} \times \frac{1}{2} = \frac{7}{3} \times \frac{1}{2} = \frac{7}{6} = 1 \frac{1}{6}$
31	ECC	$1\frac{1}{4} \times \frac{1}{3} = \frac{5}{4} \times \frac{1}{3} = \frac{5}{12}$
32	ECE	$2\frac{1}{3} \times \frac{2}{3} = \frac{7}{3} \times \frac{2}{3} = \frac{14}{9} = 1 \frac{5}{9}$
33	EDE	$1\frac{1}{4} \times \frac{3}{2} = \frac{5}{4} \times \frac{3}{2} = \frac{15}{8} = 1 \frac{7}{8}$
34	EEE	$2\frac{1}{2} \times 1\frac{2}{3} = \frac{5}{2} \times \frac{5}{3} = \frac{25}{6} = 4 \frac{1}{6}$

Table 1 (con't)

*E - mixed numeral

A - integer

D - improper fraction

C - non-unit fraction

B - unit fraction

were tested during the spring of 1976. The total number of subjects was 225. The tests were administered by the classroom teachers and required about 30 minutes for completion. They were not timed tests. Of the 225 subjects, 61 failed to meet the criteria levels established for the test of Prerequisites; 35 others were omitted because of incomplete data, and two were omitted because of perfect or zero scores. The resulting sample size was 127.

Analysis & Results

The derivation and validation of the hierarchy consisted of five steps:

Step 1. The items of the Test of Rational Number Multiplication were ranked according to difficulty. The ranking of these thirty-four tasks is presented in Table 2.

Step 2. Task and intraconcept analyses were used to hypothesize a linear hierarchy. This work was used to modify the ranking of Step 1. This analysis was completed by inspection of the means. The Step 1 arrangement, was altered to conform to the following questions. The linear sequence yielded is presented in Figure ____.

- Q 1: Is there a mixed numeral factor?
- Q 2: Is there a mixed numeral product?
- Q 3: Is there a mixed numeral factor in the right position?
(left position more difficult)
- Q 4: When one factor is a mixed numeral, is the second factor an integer, improper fraction, non-unit fraction; or unit fraction?
- Q 5: Is there an integer, improper, non-unit, or unit fraction as a factor? (Integer most difficult)
- Q 6: Is the factor of Question 5 in a right most position?
(Most difficult is right)
- Q 7: Does the factor pair of question 6 contain improper, non-unit, or unit fraction pairwise? (Improper most difficult)

The relations of the resulting hierarchy can be described in terms of two sets of nestings, one from questions 1, 2, 3, 4, and one from questions 5, 6, and 7. The resulting linear hierarchy is presented in Figure 1.

Table 2.

Linear Ordering of Items by Difficulty

Rank	Item Number	Item Difficulty
1	6	.24
2	28	.26
3	32	.27
4	30	.31
5	14	.32
6	21	.32
7	27	.32
8	33	.32
9	34	.32
10	29	.40
11	31	.41
12	22	.41
13	20	.41
14	5	.41
15	13	.43
16	8	.47
17	2	.48
18	4	.49
19	26	.57
20	19	.59
21	15	.61
22	12	.63
23	3	.64
24	7	.69
25	25	.70
26	1	.71
27	18	.84
28	9	.87
29	10	.87
30	16	.87
31	23	.87
32	17	.89
33	11	.91
34	24	.91

HIERARCHY BASED ON QUESTIONS

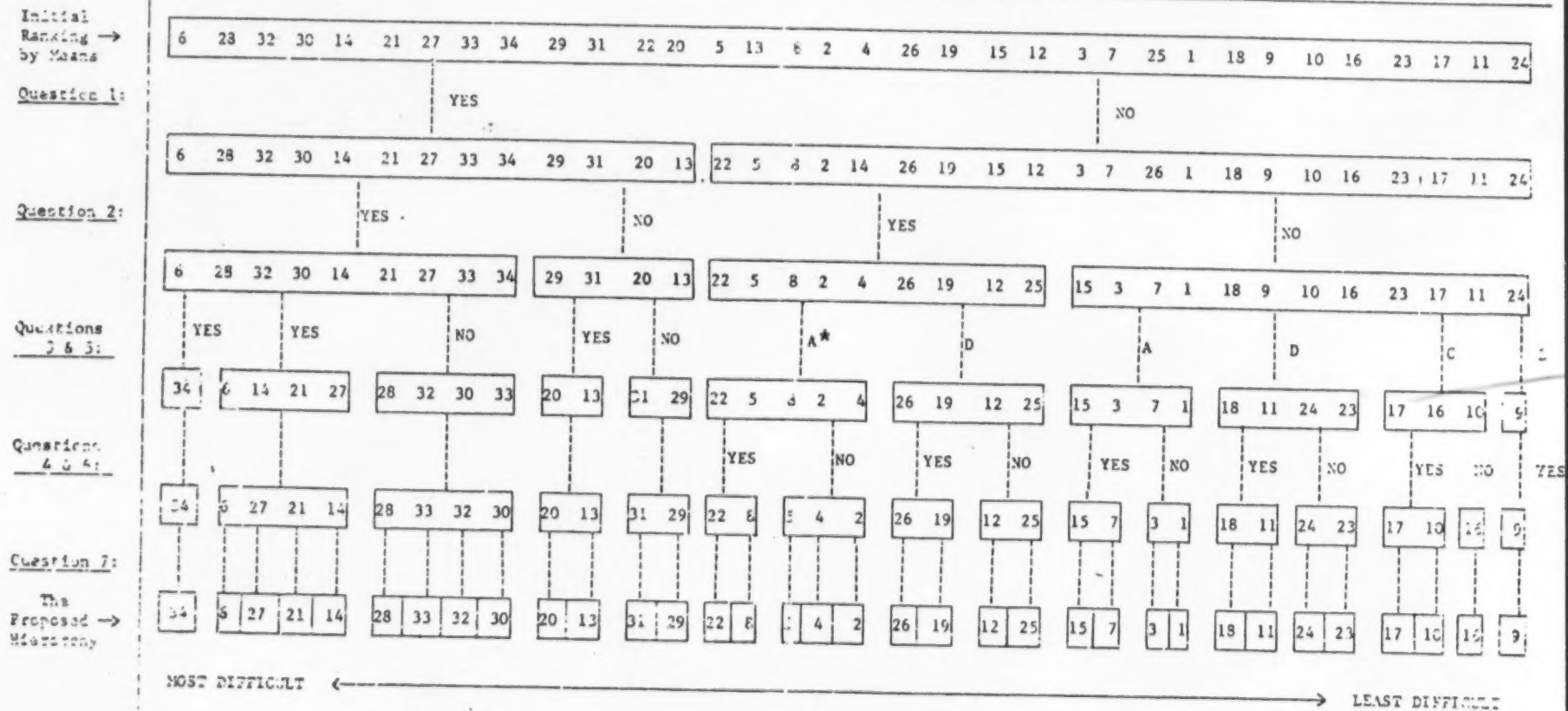


FIGURE 1

A - Integer

B - Unit Fraction

C - Non-Unit Fraction

D - Improper Fraction

Step III. The linear hierarchy hypothesized in Step II (See Figure 1) was examined using Walbesser ratios. The levels of acceptability determined by Phillips (1971) were used instead of those proposed by Walbesser (1968) since no instructional sequences were involved. Those levels are: (a) Consistency ratio = .85, (b) Adequacy ratio = .70, and Completeness ratio = .50. The hypothesized linear hierarchy was not supported. Walbesser ratios were below acceptable levels (See Table 3). The rejection of a linear hierarchy was followed by a search for branched hierarchies.

Step IV. A further examination of task difficulty, concepts, and Walbesser ratios yielded the formation of clusters in the hierarchy developed in Step II. Based on the observations, branched hierarchies were hypothesized.

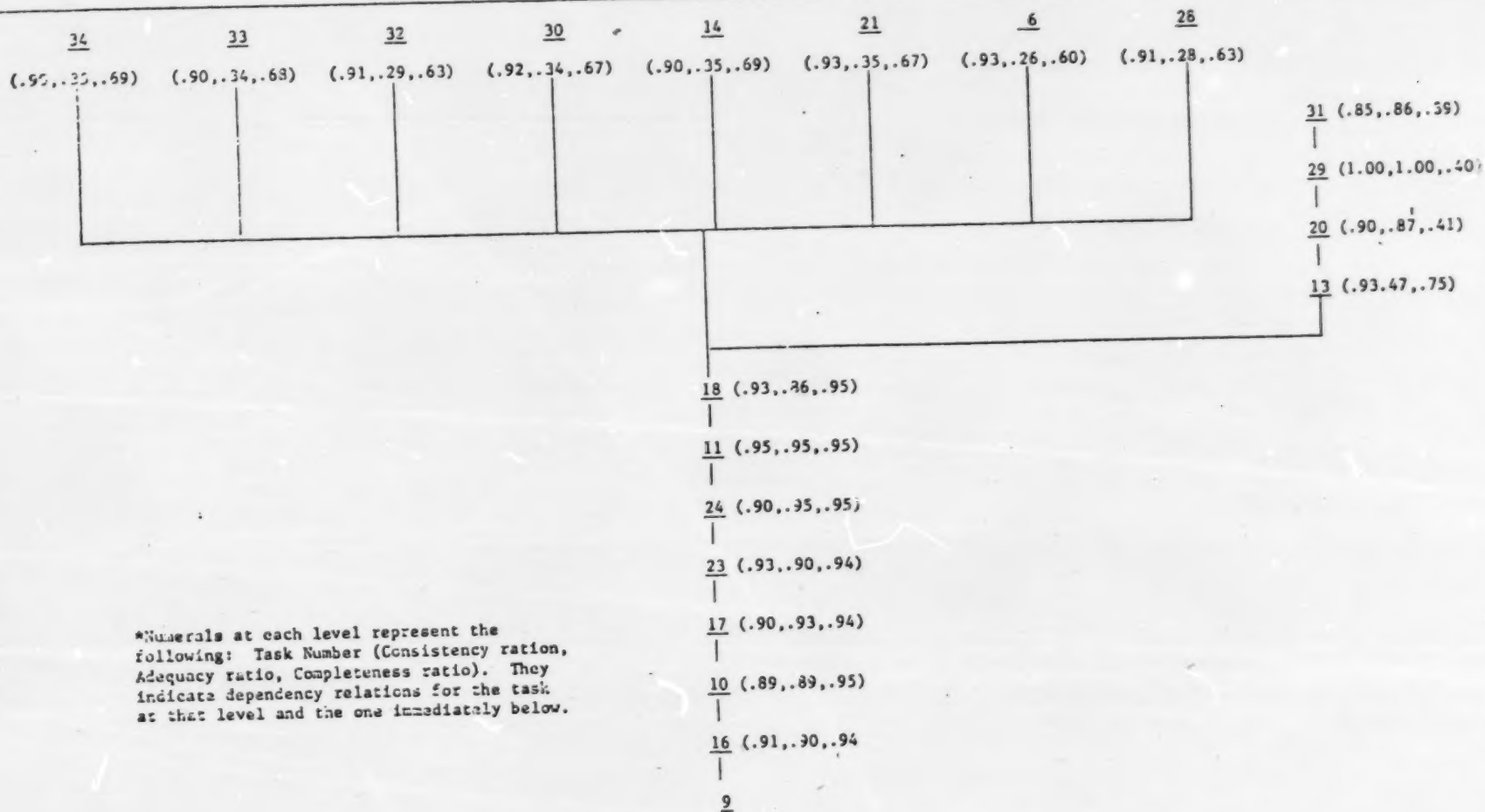
Step V. The branched hierarchies were examined using Walbesser ratios. Dependency relations were examined within and between clusters. The empirically determined hierarchies which resulted are presented in Figures 2 and 3. A discussion of dominant concepts within each cluster and dependencies among cluster members follows.

TABLE 3

WALBESSER RATIOS FOR THE HYPOTHESIZED HIERARCHY (N=127)

Task High--Low	Consistency	Adequacy	Completeness
34 -- 6	.51	.70	.21
6 -- 27	.73	.55	.22
27 -- 21	.78	.78	.28
21 -- 14	.83	.80	.29
14 -- 28	.56	.70	.23
28 -- 33	.73	.60	.24
33 -- 32	.73	.85	.26
32 -- 30	.82	.72	.25
30 -- 20	.82	.62	.32
20 -- 13	.90	.87	.41
13 -- 31	.85	.88	.41
31 -- 29	.85	.86	.39
29 -- 22	.49	.48	.34
22 -- 8	.79	.68	.42
8 -- 5	.68	.79	.42
5 -- 4	.85	.71	.44
4 -- 2	.84	.85	.48
2 -- 26	.72	.61	.54
26 -- 19	.74	.72	.61
19 -- 12	.57	.81	.49
12 -- 25	.89	.80	.71
25 -- 15	.63	.73	.77
15 -- 7	.95	.84	.67
7 -- 3	.87	.94	.68
3 -- 1	.96	.87	.70
1 -- 18	.88	.75	.89
18 -- 11	.93	.86	.95
11 -- 24	.95	.95	.95
24 -- 23	.90	.96	.96
23 -- 17	.93	.90	.94
17 -- 10	.89	.92	.95
10 -- 16	.89	.89	.95
16 -- 9	.91	.90	.94

EMPIRICAL HIERARCHY*



*Numerals at each level represent the following: Task Number (Consistency ratio, Adequacy ratio, Completeness ratio). They indicate dependency relations for the task at that level and the one immediately below.

FIGURE 2

EMPIRICAL HIERARCHY

<u>22</u> (.94,.63,.51)	<u>8</u> (.86,.68,.55)	<u>5</u> (.85,.71,.44)	<u>26</u> (.86,.70,.69)	<u>19</u> (.82,.76,.64)
		<u>4</u> (.84,.85,.48)	<u>25</u> (.63,.73,.77)	<u>12</u> (.90,.68,.85)
		<u>2</u> (.85,.69,.56)		

15 (.95,.84,.67)
 |
7 (.87,.94,.68)
 |
3 (.96,.87,.70)
 |
1 (.88,.75,.89)
 |
18 (.93,.86,.95)
 |
11 (.95,.95,.95)
 |
24 (.90,.95,.95)
 |
23 (.93,.90,.94)
 |
17 (.90,.93,.94)
 |
10 (.89,.89,.89)
 |
16 (.91,.90,.94)
 |
9

FIGURE 3

Cluster I - Tasks involving a mixed numeral (34, 33, 32, 30, 14, 21, 6, and 28) as a factor and as a product are members of this group. The means of these tasks are less than or equal to .32. Dependency relations were not exhibited among the tasks of this cluster. Unlike any other cluster tasks involved changing mixed numerals to improper fractions and improper fractions to mixed numerals.

The consistency and completeness ratios between this cluster and cluster VI were at acceptable levels (.85 and .50 respectively) but the adequacy ratio was below acceptable level (.70). The reason may be that one or more tasks were not included in the initial development to the thirty-four items.

Cluster II - Tasks of this cluster (31, 29, 20, and 13) involved a mixed numeral factor and a non-unit fraction product. Their difficulty ranged from .40 to .43. Members of this cluster exhibited dependency relations as hypothesized.

Dependency relations were supported between this cluster and cluster VI. The consistency and adequacy ratios were at acceptable levels while the completeness ratio was not for tasks 31, 29, and 20. The low adequacy ratio is an indication of a need for additional instruction. The remaining task of this cluster, task 13, had acceptable levels for consistency and completeness ratios but a low adequacy ratio (.47). Again, it may be because of the omission of a certain task.

Cluster III - This cluster (22, 8, 5, 4, and 2) involved tasks with an integer as a factor and a mixed numeral as a product. A linear hierarchy was exhibited among three of the cluster's items (5, 4, 2) respectively. These three items have an integer factor in the first position and are ordered according to second position component - improper, non-unit and unit fraction respectively. Dependency relations were supported between

this cluster and cluster V.

Tasks 8, 26, and 19 had acceptable levels for the three Walbesser ratios. Other cluster members are below level in one of the ratios. Tasks 22, 2, and 12 are below adequacy level (.70) by less than .1. Tasks 5 and 4 are below completeness level (.50) by less than .1. These are very close to acceptable level. Task 25 is below acceptable level (.85) by .22.

Cluster IV - Tasks of this cluster (26, 25, 19, and 12) did not have a mixed numeral as a factor but as a product. The group means ranged from .41 to .49. Dependency relations are exhibited between tasks 26 and 25, and tasks 19 and 12 respectively. Items 26 and 25 both have an improper fraction as first factor and ordered according to second factor components - improper and non-unit respectively. Items 19 and 12 have an improper fraction as second factor and ordered according to first factor components - non-unit and unit fraction respectively.

Cluster IV is not related to clusters I, II, and III as discussed above. It is related to cluster V as hypothesized.

Cluster V - Tasks of this cluster (15, 7, 3, and 1) involved an integer as one of the factors and a non-unit fraction as the product. The group means ranged from .61 to .71. As hypothesized, linear relationships were exhibited among the members of this cluster and between this cluster and cluster VI. Clusters I, II, III, and IV have been discussed relative to cluster V.

Cluster VI - Tasks of this cluster (18, 11, 24, 23, 17, 16, and 10) did not involve a mixed numeral or an integer as a factor. The group means ranged from .84 to .91. All products were non-unit fractions. Tasks of this cluster exhibited linear relationships as hypothesized. This cluster is related to cluster VII.

Cluster VII - This cluster of one item (9) involved two unit fractions as factors and a unit fraction as product. The item mean is .87.

The following linear hierarchies were supported after observing relationships among the Clusters.

1. Each task of Cluster 1 with tasks of Cluster 6. Eight hierarchies resulted.
2. The tasks of Clusters 2 and 6 formed a linear hierarchy.
3. The tasks of Clusters 4, 5, and 6 formed three hierarchies:
(a) Task 22 with Clusters 5 and 6, (b) Task 8 with Clusters 5 and 6, (c) Tasks 5, 4, and 2 with Clusters 5 and 6.
4. Tasks of Clusters 3, 5, and 6 resulted in two hierarchies:
(a) Tasks 26 and 25 with Clusters 5 and 6, (b) Tasks 19 and 12 with Clusters 5 and 6.

A total of fourteen hierarchies resulted from the analyses of the data.

Discussion

This is the fourth study which demonstrates that hierarchies can be generated from analyses of performance data collected from subjects at various grade levels; the data are collected from students who have been taught the superordinate task one or more times in the regular curriculum.

Since the hierarchies in this and the three previous studies by Hill and Uprichard and Phillips were modified based on subjects' performances, it is inappropriate to say that these studies are validation studies; it is more appropriate to say that they are hierarchy generation studies. The hierarchies yielded in each of these studies should be validated by evaluating the performance of new samples on the hierarchies now obtained.

A third point is that the hierarchies are probably not learning hierarchies. Learning hierarchies have been used to refer to sequences for optimal learning of superordinate tasks. Since the hierarchies yielded in these three studies were generated on performance data from subjects who have already encountered instruction on the super ordinate task in the regular curriculum, the hierarchy might more appropriately be labeled a retention hierarchy. This is particularly important in view of Gagne and Bassler's (1963) study which concluded that once superordinate tasks are learned, selective retention and forgetting of prerequisite learning subtasks occurs. However, the hierarchies generated are of particular interest to research studies in diagnosis and remediation since these hierarchies may provide the data necessary for reteaching rather than original instruction.

The final point is relative to statistical tests for examining hierarchies of this type. Phillips proposed changes in criterion levels for the Walbesser ratios; since no test of significance exists, the values selected may be unacceptably high or low.

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